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Functional representation of the Ablowitz–Ladik hierarchy

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Abstract. The Ablowitz–Ladik hierarchy (ALH) is considered in the framework of the inverse-scattering approach. After establishing the structure of solutions of the auxiliary linear problems, the ALH, which has been originally introduced as an infinite system of differential-difference equations, is presented as a finite system of difference-functional equations. The representation obtained, when rewritten in terms of Hirota's bilinear formalism, is used to demonstrate relations between the ALH and some other integrable systems, the Kadomtsev–Petviashvili hierarchy in particular.

1. Introduction

Among various concepts of the theory of integrable nonlinear systems one of the most fruitful is a viewpoint when each integrable equation is considered as member of an infinite number of related equations—hierarchy [1]. So, for example the famous nonlinear Schrödinger equation is the simplest equation of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, its discrete integrable analogue is a member of the Ablowitz–Ladik hierarchy (ALH), etc. A distinguishing feature of integrable hierarchies is that corresponding flows commute, i.e. all its equations are compatible. This enables us to consider a hierarchy as one system of equations, i.e. to consider an infinite number of, say, $(1 + 1)$-dimensional partial differential equations (PDEs) (as, for example, in the case of the AKNS hierarchy) or differential-difference equations (DDEs) (in the case of the ALH) as a $(1+\infty)$-dimensional problem for functions depending on an infinite number of variables. Such an approach has been intensively studied for almost all 'classical' integrable equations and has been shown to be a rather powerful tool for tackling integrable nonlinear problems. A logical continuation of this method is to 'convert' this infinite number of PDEs or DDEs into one or several functional equations which relate functions taken at different values of its arguments, say, z and $z \pm \varphi(\lambda)$ where λ is some auxiliary parameter (in some contexts such equations have been called 'addition formulae', in soliton theory they are also known as 'bilinear identities for τ-functions'). These functional equations can be viewed as generating functions for the hierarchy considered: expanding them in power series in λ one can obtain all equations of the hierarchy. In what follows I will use the term 'functional' for such a representation, bearing in mind that we are dealing with the *functional* equations which are equivalent to (i.e. represent) a hierarchy of PDEs or DDEs. Such functional equations naturally appear in the Sato theory of soliton equations [2]. Also, this question, especially in the case of the Kadomtsev–Petviashvili (KP) hierarchy, has been discussed in connection with the problem of characterization of the Jacobi varieties (see, e.g. [3, 4] and references therein). Some

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recent examples of the functional representation of integrable systems one can find, for example, in [5–8] (KP and dispersionless KP hierarchies) and [9, 10] (Toda hierarchy).

In this paper the case of the ALH will be considered. After outlining some basic facts related to the inverse scattering transform (IST) (section 2) and discussing more comprehensively corresponding linear problems (section 3) the functional representation of the ALH will be obtained (section 4). In section 5 the results obtained will be rewritten in terms of Hirota's bilinear operators. This will expose some relations between the ALH and other integrable hierarchies, KP hierarchy in particular.

2. Zero curvature representation of the ALH

The ALH is an infinite set of ordinal DDEs, that has been introduced by Ablowitz and Ladik in 1975 [11]. The most well known of these equations is the discrete nonlinear Schrodinger equation (DNLSE)

$$\mathrm{i}\dot{q}_n = q_{n+1} - 2q_n + q_{n-1} - q_n r_n \left(q_{n+1} + q_{n-1}\right) \tag{1}$$

and the discrete modified KdV equation (DMKdV),

$$\dot{q}_n = p_n(q_{n+1} - q_{n-1}) \tag{2}$$

where

$$p_n = 1 - q_n r_n \qquad r_n = -\kappa\bar{q}_n \qquad \kappa = \pm 1 \tag{3}$$

(see e.g. [12]). All equations of the ALH can be presented as the compatibility condition for the linear system

$$\Psi_{n+1} = U_n \Psi_n \tag{4}$$

$$\partial_t \Psi_n = V_n \Psi_n \tag{5}$$

where ∂_t stands for $\partial/\partial t$, which leads to their zero-curvature representation (ZCR):

$$\partial_t U_n = V_{n+1} U_n - U_n V_n. \tag{6}$$

In the standard IST approach developed in [11] the matrix U_n for the ALH is given by

$$U_n = \begin{pmatrix} \lambda & r_n \\ q_n & \lambda^{-1} \end{pmatrix} \tag{7}$$

where λ is the auxiliary constant parameter. For the elements of the matrix V_n,

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{8}$$

one can then obtain from (6), the system of equations

$$\lambda(a_{n+1} - a_n) = -q_n b_{n+1} + r_n c_n \tag{9}$$

$$\lambda^{-1}(d_{n+1} - d_n) = q_n b_n - r_n c_{n+1} \tag{10}$$

$$\partial_t q_n = q_n(d_{n+1} - a_n) + \lambda c_{n+1} - \lambda^{-1} c_n \tag{11}$$

$$\partial_t r_n = r_n(a_{n+1} - d_n) - \lambda b_n + \lambda^{-1} b_{n+1}. \tag{12}$$

According to [11], they can be chosen as Laurent polynomials in λ in such a way that (9)–(12) hold automatically for all λ's provided the q_n's and r_n's satisfy some differential relations. It should be noted that one can obtain an infinite number of the matrices V_n (which are Laurent polynomials of different order) which leads to the infinite number of differential equations $\partial q_n/\partial t = F_n^l$, $(l = 1, 2, \ldots)$. According to the now widely accepted viewpoint, as mentioned in the introduction, one can consider the q_n's and r_n's as depending on the infinite number of 'times', $q_n = q_n(t_1, t_2, \ldots)$ and consider the lth equation of the ALH as describing the flow with respect to the lth variable, $\partial q_n/\partial t_l = F_n^l$. I will also adhere to the conception of q_n's being functions of the infinite number of variables, but my approach will differ slightly from the classical one in the following aspect. Traditionally it is implied that all 'times' t_l are real, which is grounded from the standpoint of physical applications, and is convenient in the framework of the inverse-scattering technique. I will use, instead of real 'times' t_l, some complex variables $z_j, \bar{z}_j$ $(j = 1, 2, \ldots)$, which, as will be shown below, exhibit in a more transparent way some intrinsic properties of the ALH. A simple analysis yields that the family of possible solutions of the system (9)–(12) (and hence the equations of the hierarchy) can be divided into two subsystems. One of them consists of V-matrices which are polynomials in λ^{-1} (I will term the corresponding equations as a 'positive' part of hierarchy) and the other consist of matrices which are polynomials in λ ('negative' subhierarchy) while in the standard 'real-time' approach all the V-matrices contain terms proportional to λ^m together with the terms proportional to λ^{-m} $(m \geqslant 0)$. Let us consider first the 'positive' case. An infinite number of polynomial in $1/\lambda$ solutions V_n^j $(j = 1, 2, \ldots)$ possesses the following structure:

$$V_n^j = \lambda^{-2} V_n^{j-1} + \begin{pmatrix} \lambda^{-2}\alpha_n^j & \lambda^{-1}\beta_n^j \\ \lambda^{-1}\gamma_n^j & \delta_n^j \end{pmatrix} \tag{13}$$

where the elements $\alpha_n^j, \ldots, \delta_n^j$ satisfy the equations

$$\alpha_{n+1}^j - \alpha_n^j = -q_n\beta_{n+1}^j + r_n\gamma_n^j \tag{14}$$

$$\delta_{n+1}^j - \delta_n^j = q_n\beta_n^j - r_n\gamma_{n+1}^j \tag{15}$$

$$\partial_j q_n = q_n\delta_{n+1}^j + \gamma_{n+1}^j = q_n\alpha_n^{j+1} + \gamma_n^{j+1} \tag{16}$$

$$\partial_j r_n = -r_n\delta_n^j - \beta_n^j = -r_n\alpha_{n+1}^{j+1} - \beta_{n+1}^{j+1} \tag{17}$$

with $\partial_j = \partial/\partial z_j$. Rewriting this system as

$$\alpha_n^j = -\delta_n^{j-1} \tag{18}$$

$$\beta_n^j = \beta_{n-1}^{j-1} + r_{n-1}(\delta_{n-1}^{j-1} + \delta_n^{j-1}) \tag{19}$$

$$\gamma_n^j = \gamma_{n+1}^{j-1} + q_n(\delta_n^{j-1} + \delta_{n+1}^{j-1}) \tag{20}$$

$$\delta_n^j - \delta_{n+1}^j = -q_n\beta_n^j + r_n\gamma_{n+1}^j \tag{21}$$

and choosing

$$a_n^0 = 0 \qquad b_n^0 = 0 \qquad c_n^0 = 0 \qquad d_n^0 = -\mathrm{i} \tag{22}$$

we can consequently obtain

$$\begin{aligned} &\alpha_n^1 = 0 && \alpha_n^2 = -\mathrm{i}r_{n-1}q_n \\ &\beta_n^1 = -\mathrm{i}r_{n-1} && \beta_n^2 = -\mathrm{i}r_{n-2}p_{n-1} + \mathrm{i}r_{n-1}^2 q_n \\ &\gamma_n^1 = -\mathrm{i}q_n && \gamma_n^2 = -\mathrm{i}p_n q_{n+1} + \mathrm{i}r_{n-1}q_n^2 \\ &\delta_n^1 = \mathrm{i}r_{n-1}q_n && \delta_n^2 = \mathrm{i}r_{n-2}p_{n-1}q_n + \mathrm{i}r_{n-1}p_n q_{n+1} - \mathrm{i}r_{n-1}^2 q_n^2 \end{aligned} \tag{23}$$

and, in principle, all other matrices V_n^j. This leads to the infinite system of equations for q_n, r_n, some of the first are

$$\partial_1 q_n = -\mathrm{i}p_n q_{n+1} \tag{24}$$

$$\partial_1 r_n = \mathrm{i}r_{n-1}p_n \tag{25}$$

$$\partial_2 q_n = \mathrm{i} r_{n-1} p_n q_n q_{n+1} + \mathrm{i} p_n r_n q_{n+1}^2 - \mathrm{i} p_n p_{n+1} q_{n+2} \tag{26}$$

$$\partial_2 r_n = \mathrm{i} r_{n-2} p_{n-1} p_n - \mathrm{i} r_{n-1}^2 p_n q_n - \mathrm{i} r_{n-1} p_n r_n q_{n+1}. \tag{27}$$

Analogously, looking for the V-matrices of the form

$$V_n^{-j} = \lambda^2 V_n^{-j+1} + \begin{pmatrix} \alpha_n^{-j} & \lambda\beta_n^{-j} \\ \lambda\gamma_n^{-j} & \lambda^2\delta_n^{-j} \end{pmatrix} \tag{28}$$

and repeating the procedure described above one can obtain the 'negative' part of the ALH. Some of the first of its equations are

$$\partial_{-1} q_n = -\mathrm{i} q_{n-1} p_n \tag{29}$$

$$\partial_{-1} r_n = \mathrm{i} p_n r_{n+1} \tag{30}$$

$$\partial_{-2} q_n = -\mathrm{i} q_{n-2} p_{n-1} p_n + \mathrm{i} q_{n-1} p_n q_n r_{n+1} + \mathrm{i} q_{n-1}^2 p_n r_n \tag{31}$$

$$\partial_{-2} r_n = -\mathrm{i} q_{n-1} p_n r_n r_{n+1} - \mathrm{i} p_n q_n r_{n+1}^2 + \mathrm{i} p_n p_{n+1} r_{n+2} \tag{32}$$

where $\partial_{-j} = \partial/\partial\bar{z}_j$ and the overbar denotes the complex conjugation.

Before proceeding further I would like to note that the simplest equations of the ALH, (24) and (29), when rewritten in terms of the real variables $x = \mathrm{Re}\, z_1$ and $y = \mathrm{Im}\, z_1$ become exactly the DNLSE (1) modified by the substitution $q_n \to q_n \exp(2\mathrm{i}x)$ and the DMKdV (2).

All equations of the ALH, as well as all equations of other integrable hierarchies, can be presented in bilinear form using Hirota's operators

$$D_x^a \ldots D_y^b u \cdot v = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^a \ldots \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^b u(x, y, \ldots) v(x', y', \ldots)\bigg|_{x'=x, y'=y, \ldots}. \tag{33}$$

To this end consider the functions τ_n, σ_n and ρ_n defined by

$$p_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \qquad q_n = \frac{\sigma_n}{\tau_n} \qquad r_n = \frac{\rho_n}{\tau_n}. \tag{34}$$

Note that originally we had two independent functions, q_n and r_n, for given n. The quantity p_n defined in (3) has been introduced only for the sake of shortening the formulae. Now we have three sets of functions (τ_n, σ_n, ρ_n for $n = 0, \pm 1, \ldots$). However, they are related by the equation $\tau_{n-1}\tau_{n+1} = \tau_n^2 - \sigma_n\rho_n$ which is the definition of p_n rewritten in bilinear form and which will repeatedly appear in the following consideration (see (83) and (93) below). The first equations of the ALH, (24) and (29), can then be rewritten, using the designation

$$D_j = D_{z_j} \qquad \bar{D}_j = D_{\bar{z}_j} \tag{35}$$

as

$$D_1 \sigma_n \cdot \tau_n = -\mathrm{i}\sigma_{n+1}\tau_{n-1} \tag{36}$$

$$\bar{D}_1 \sigma_n \cdot \tau_n = -\mathrm{i}\sigma_{n-1}\tau_{n+1}. \tag{37}$$

(The corresponding equations for the functions ρ_n can be obtained from these equations using the involution $\sigma_n = -\kappa\bar{\rho}_n$.) The next pair of equations of the ALH, (26) and (31), can be presented as

$$D_2 \sigma_n \cdot \tau_n = D_1 \sigma_{n+1} \cdot \tau_{n-1} \tag{38}$$

$$D_1 \tau_{n+1} \cdot \tau_n = \mathrm{i}\sigma_{n+1}\rho_n \tag{39}$$

and

$$\bar{D}_2 \sigma_n \cdot \tau_n = \bar{D}_1 \sigma_{n-1} \cdot \tau_{n+1} \tag{40}$$

$$\bar{D}_1 \tau_{n+1} \cdot \tau_n = -\mathrm{i}\sigma_n\rho_{n+1}. \tag{41}$$

The bilinear representation of the higher equations of the hierarchy will be discussed below, and here I would like to mention only one remarkable fact. By simple calculations one can obtain an alternative representation for the equations (38) and (39):

$$(\mathrm{i}D_2 + D_{11})\sigma_n \cdot \tau_n = 0 \tag{42}$$

(hereafter I will write $D_{x\ldots y}$ instead of $D_x \ldots D_y$) which involves functions for only one value of the index n. In other words we have presented the differential-*difference* equations (38) and (39) in a form typical to *partial* differential equation. Obviously, equation (42) taken alone cannot be considered as a closed PDE system, to be such it must be complemented with some other relations involving σ and τ, which can be achieved using other equations of the ALH (see e.g. (105) below). Analogously one can rewrite equations (40) and (41) as well as all other equations of the ALH. It is a manifestation of the fact that both 'positive' and 'negative' subhierarchies can be transformed into hierarchies of $(1+1)$-dimensional evolution equations for $q = q_n$ and $r = r_n$ as functions of $z = z_1$ and z_j, $j = 2, 3, \ldots$. Indeed, expressing from (24) and (25) q_{n+1}, q_{n+2} and r_{n-1} in terms of q, q_z, q_{zz}, r, r_z (here the subscripts indicate derivatives with respect to z) equations (26) and (27) can be rewritten as

$$\mathrm{i}\partial_2 q + q_{zz} + \frac{2qq_z r_z}{1-qr} = 0 \tag{43}$$

$$-\mathrm{i}\partial_2 r + r_{zz} + \frac{2rr_z q_z}{1-qr} = 0. \tag{44}$$

All higher equations of the 'positive' hierarchy can be rewritten in a similar way. An analogous procedure can be performed for the 'negative' part of the hierarchy. Some general formulae for such a representation of the ALH will be obtained in section 5.

3. Solutions of the auxiliary problems

Now some solutions of the linear problems (4) and (5), which I rewrite now as,

$$\partial_j \Psi_n = V_n^j \Psi_n \tag{45}$$

will be constructed.

The question of solving (4) and (45) is not the main aim of this paper, but it is discussed for illustrative purposes, to show how one can 'deduce' the functional representation of the ALH which we are looking for, which can then be proved independently, without invoking the results of this section. That is why some results (namely formulae (62) and (64)) are written down without a presentation of their rigorous proof.

In what follows I will restrict myself to the simplest case

$$\lim_{n\to\infty} q_n, r_n = 0 \tag{46}$$

or, in the $\{\tau, \sigma, \rho\}$-representation,

$$\lim_{n\to\infty} \sigma_n, \rho_n = 0 \qquad \lim_{n\to\infty} \tau_n = \text{constant}. \tag{47}$$

Presenting the elements of the first column of the matrix Ψ_n as

$$\Psi_n^{(1)} = \lambda^n \frac{\tau_n}{\tau_{n-1}} \begin{pmatrix} \varphi_n \\ -\lambda\psi_n \end{pmatrix} \tag{48}$$

one can obtain from (4) the following equations for the quantities φ_n, ψ_n:

$$p_n \varphi_{n+1} = \varphi_n - r_n \psi_n \tag{49}$$

$$-\lambda^2 p_n \psi_{n+1} = q_n \varphi_n - \psi_n \tag{50}$$

which will be solved under the boundary conditions

$$\lim_{n\to\infty} \varphi_n = 1 \qquad \lim_{n\to\infty} \psi_n = 0. \tag{51}$$

This problem admits solutions that can be presented as power series in λ^2:

$$\begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = \sum_{m=0}^{\infty} \lambda^{2m} \begin{pmatrix} \varphi_n^m \\ \psi_n^m \end{pmatrix}. \tag{52}$$

Substituting these series in (49) and (50) one can derive a system of equations for the quantities φ_n^m and ψ_n^m, which can be written as follows:

$$\varphi_n^m - \varphi_{n-1}^m = -r_{n-1}\psi_n^{m-1} \tag{53}$$

$$\psi_n^m = q_n\varphi_n^m + p_n\psi_{n+1}^{m-1}. \tag{54}$$

From this system and (51) one can obtain

$$\varphi_n^0 = 1 \qquad \psi_n^0 = q_n. \tag{55}$$

Using the identity

$$\partial_1 \ln \frac{\tau_n}{\tau_{n-1}} = \mathrm{i} r_{n-1} q_n \tag{56}$$

which follows from (39) one can perform an iteration:

$$\varphi_n^1 = \frac{\mathrm{i}}{\tau_n}\partial_1\tau_n \qquad \psi_n^1 = \frac{\mathrm{i}}{\tau_n}\partial_1\sigma_n. \tag{57}$$

Further, using

$$\partial_2 \ln \frac{\tau_n}{\tau_{n-1}} = \mathrm{i} r_{n-2} p_{n-1} q_n + \mathrm{i} r_{n-1} p_n q_{n+1} - \mathrm{i} r_{n-1}^2 q_n^2 \tag{58}$$

one can obtain

$$\varphi_n^2 = \frac{1}{2\tau_n}(\mathrm{i}\partial_2 - \partial_{11})\tau_n \qquad \psi_n^2 = \frac{1}{2\tau_n}(\mathrm{i}\partial_2 - \partial_{11})\sigma_n. \tag{59}$$

Iterating the system (53) and (54) further one can conclude that the quantities $\tau_n\varphi_n^m$ and $\tau_n\psi_n^m$ are the coefficients of the Taylor expansion for the functions $\tau_n(z_1+\mathrm{i}\lambda^2, z_2+\mathrm{i}\lambda^4/2, \ldots)$ and $\sigma_n(z_1 + \mathrm{i}\lambda^2, z_2 + \mathrm{i}\lambda^4/2, \ldots)$. Moreover, it can be shown that the column (48) with

$$\varphi_n = \frac{\tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)}{\tau_n(z_k, \bar{z}_k)} \qquad \psi_n = \frac{\sigma_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)}{\tau_n(z_k, \bar{z}_k)} \tag{60}$$

where τ_n and σ_n are solutions of the ‘positive’ subhierarchy, solve the linear problems (45) for $j = 1, 2, \ldots$. Here the designation

$$f(z_k, \bar{z}_k) \equiv f(z_1, z_2, \ldots \bar{z}_1, \bar{z}_2, \ldots) \tag{61}$$

is used.

Considering in a similar way the second column of the matrix Ψ_n one can obtain the following matrix solution for the linear problems of the ‘positive’ subhierarchy (i.e. the problems (4) and (45) for $j = 1, 2, \ldots$):

$$\Psi_n^+ = \frac{1}{\tau_{n-1}(z_k, \bar{z}_k)} \begin{pmatrix} \lambda^n \tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) & \lambda^{-n+1}\exp(-\mathrm{i}\phi)\rho_{n-1}(z_k - \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \\ -\lambda^{n+1}\sigma_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) & \lambda^{-n}\exp(-\mathrm{i}\phi)\tau_{n-1}(z_k - \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \end{pmatrix} \tag{62}$$

where

$$\phi = \sum_{k=1}^{\infty} \lambda^{-2k} z_k. \tag{63}$$

Analogously, for the linear problems of the 'negative' subhierarchy, (4) and (45) for $j = -1, -2, \ldots$, one can obtain the solution

$$\Psi_n^- = \frac{1}{\tau_{n-1}(z_k, \bar{z}_k)} \begin{pmatrix} \lambda^n \exp(\mathrm{i}\tilde{\phi})\tau_{n-1}(z_k, \bar{z}_k + \mathrm{i}\lambda^{-2k}/k) & -\lambda^{-n-1}\rho_n(z_k, \bar{z}_k - \mathrm{i}\lambda^{-2k}/k) \\ \lambda^{n-1} \exp(\mathrm{i}\tilde{\phi})\sigma_{n-1}(z_k, \bar{z}_k + \mathrm{i}\lambda^{-2k}/k) & \lambda^{-n}\tau_n(z_k, \bar{z}_k - \mathrm{i}\lambda^{-2k}/k) \end{pmatrix} \tag{64}$$

where

$$\tilde{\phi} = \sum_{k=1}^{\infty} \lambda^{2k} \bar{z}_k. \tag{65}$$

The obtained formulae (62) and (64) illustrate the fact that an integrable hierarchy is more than a collection of solvable equations, and by considering a hierarchy one can sometimes obtain more 'transparent' results than by dealing with one particular equation. Such an approach (and such results) is not entirely new; it had been applied earlier to other hierarchies, say AKNS [1], though, to the author's knowledge, for the ALH this has been done for the first time in this work.

4. The main result

In section 3 we constructed the matrices Ψ_n^+ (Ψ_n^-) which are solutions of the discrete auxiliary problem (4) and the 'positive' ('negative') set of evolution linear problems (45). Though the validity of these results should be discussed more precisely, they give us a sufficient hint to obtain the main result of this work, namely, the functional representation of the ALH. The matrix equation $\Psi_{n+1}^+ = U_n \Psi_n^+$ after some transformations can be rewritten in the following way:

$$\begin{aligned} &\sigma_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) - \sigma_n(z_k, \bar{z}_k)\tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \\ &\qquad = \lambda^2 \tau_{n-1}(z_k, \bar{z}_k)\sigma_{n+1}(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \end{aligned} \tag{66}$$

$$\begin{aligned} &\rho_n(z_k, \bar{z}_k)\tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) - \rho_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) \\ &\qquad = \lambda^2 \rho_{n-1}(z_k, \bar{z}_k)\tau_{n+1}(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \end{aligned} \tag{67}$$

$$\begin{aligned} &\tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) - \tau_{n-1}(z_k, \bar{z}_k)\tau_{n+1}(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k) \\ &\qquad = \sigma_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)\rho_n(z_k, \bar{z}_k). \end{aligned} \tag{68}$$

Now we can forget about were these equations originate from and consider them as a system of three difference-functional *equations* for unknown functions τ_n, σ_n and ρ_n. This system is compatible with all 'positive' flows of the ALH: if τ_n, σ_n and ρ_n solve (66)–(68) then Ψ_n^+ satisfy all equations (45) for $j = 1, 2, \ldots$. To prove this consider the quantities

$$X_n^j = \partial_j \varphi_n - a_n^j \varphi_n + \lambda b_n^j \psi_n \tag{69}$$

$$Y_n^j = \partial_j \psi_n + \lambda^{-1} c_n^j \varphi_n - d_n^j \psi_n \tag{70}$$

where φ_n, ψ_n are defined by

$$\varphi_n = \frac{\tau_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)}{\tau_{n-1}(z_k, \bar{z}_k)} \qquad \psi_n = \frac{\sigma_n(z_k + \mathrm{i}\lambda^{2k}/k, \bar{z}_k)}{\tau_{n-1}(z_k, \bar{z}_k)}. \tag{71}$$

(Note that these functions φ_n, ψ_n differ from ones defined by (60) in the factor τ_n/τ_{n-1}) and $a_n^j, \ldots, d_n^j$ are elements of the matrix V_n^j (see (8).) Using the identities

$$\varphi_{n+1} - \varphi_n + r_n \psi_n = 0 \tag{72}$$

$$\lambda^2 \psi_{n+1} - \psi_n + q_n \varphi_n = 0 \tag{73}$$

which follow from (66) and (68) it is straightforward to verify the fact that the combination $X_{n+1} - X_n + r_n Y_n$ can be presented as

$$X_{n+1} - X_n + r_n Y_n = \partial_j[\varphi_{n+1} - \varphi_n + r_n\psi_n] + [a_n^j - a_{n+1}^j - \lambda^{-1}q_n b_{n+1}^j + \lambda^{-1}r_n c_n^j]\varphi_n$$
$$+[-\partial_j r_n + (a_{n+1}^j - d_n^j) - \lambda b_n^j + \lambda^{-1}b_{n+1}^j]\psi_n. \qquad (74)$$

It can easily be seen from (72) together with (9) and (12) that all expressions in square brackets are equal to zero, i.e.

$$X_{n+1} - X_n + r_n Y_n = 0. \qquad (75)$$

Analogously, calculating in a similar way $\lambda^2 Y_{n+1} - Y_n + q_n X_n$ one can obtain

$$\lambda^2 Y_{n+1} - Y_n + q_n X_n = 0. \qquad (76)$$

Presenting X_n^j and Y_n^j as

$$X_n^j = \frac{\tau_n}{\tau_{n-1}}\sum_{m=0}^{\infty}\lambda^{2m}X_{n,m}^j \qquad Y_n^j = \frac{\tau_n}{\tau_{n-1}}\sum_{m=0}^{\infty}\lambda^{2m}Y_{n,m}^j \qquad (77)$$

one can derive the following recurrence for the coefficients $X_{n,m}^j$, $Y_{n,m}^j$:

$$X_{n,m}^j - X_{n-1,m}^j = -\frac{\rho_{n-1}}{\tau_n}Y_{n,m-1}^j \qquad (78)$$

$$Y_{n,m}^j = p_n Y_{n+1,m-1}^j + q_n X_{n,m}^j. \qquad (79)$$

It can be shown that $X_n(\lambda = 0) = Y_n(\lambda = 0) = 0$, i.e. $X_{n,0}^j = Y_{n,0}^j = 0$. Then, (78) yields $X_{n,1}^j =$ constant, this constant, as follows from the boundary conditions for φ_n and ψ_n, is zero, $X_{n,1}^j = 0$. This in turn, together with (79), leads to $Y_{n,1}^j = 0$, etc. Repeating iterations one can obtain $X_n^j = Y_n^j = 0$, which implies that the column $(\varphi_n, -\lambda\psi_n)^T$ is a solution of the equation

$$\partial_j\begin{pmatrix}\varphi_n\\ -\lambda\psi_n\end{pmatrix} = V_n^j\begin{pmatrix}\varphi_n\\ -\lambda\psi_n\end{pmatrix}. \qquad (80)$$

Thus, we have shown that equations (66)–(68), which I would like to rewrite in a more symmetrical way using the substitutions $z_k \to z_k \mp \mathrm{i}\lambda^{2k}/2k$,

$$\sigma_n(z_k + \mathrm{i}\delta^k/2k)\tau_n(z_k - \mathrm{i}\delta^k/2k) - \sigma_n(z_k - \mathrm{i}\delta^k/2k)\tau_n(z_k + \mathrm{i}\delta^k/2k)$$
$$= \delta\tau_{n-1}(z_k - \mathrm{i}\delta^k/2k)\sigma_{n+1}(z_k + \mathrm{i}\delta^k/2k) \qquad (81)$$

$$\rho_n(z_k - \mathrm{i}\delta^k/2k)\tau_n(z_k + \mathrm{i}\delta^k/2k) - \rho_n(z_k + \mathrm{i}\delta^k/2k)\tau_n(z_k - \mathrm{i}\delta^k/2k)$$
$$= \delta\rho_{n-1}(z_k - \mathrm{i}\delta^k/2k)\tau_{n+1}(z_k + \mathrm{i}\delta^k/2k) \qquad (82)$$

$$\tau_n(z_k + \mathrm{i}\delta^k/2k)\tau_n(z_k - \mathrm{i}\delta^k/2k) - \tau_{n-1}(z_k - \mathrm{i}\delta^k/2k)\tau_{n+1}(z_k + \mathrm{i}\delta^k/2k)$$
$$= \sigma_n(z_k + \mathrm{i}\delta^k/2k)\rho_n(z_k - \mathrm{i}\delta^k/2k) \qquad (83)$$

(here δ is used instead of λ^2 and dependence on the conjugated coordinates, $\bar{z}_k$, is temporarily omitted) are indeed compatible with the 'positive' flows (45) for $j = 1, 2, \ldots$, and can be considered as being equivalent to the 'positive' part of the ALH. Expanding (81)–(83) in powers of δ an infinite number of DDEs, that can be transformed to those from the ALH, can be obtained. Thus, for example, the equations which correspond to the first power of δ

$$(\partial_1\sigma_n)\tau_n - \sigma_n(\partial_1\tau_n) = -\mathrm{i}\tau_{n-1}\sigma_{n+1} \qquad (84)$$

$$(\partial_1\rho_n)\tau_n - \rho_n(\partial_1\tau_n) = \mathrm{i}\rho_{n-1}\tau_{n+1} \qquad (85)$$

are obviously the equations (24) and (25) rewritten in the $\{\tau_n, \sigma_n, \rho_n\}$-representation. The equations which correspond to the second power of δ are equivalent to the second pair of equations of the ALH, (26) and (27), etc.

Analogously, the 'negative' part of the ALH can be written as the following functional equations:

$$\sigma_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\tau_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) - \sigma_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k)\tau_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k) = \tilde{\delta}\sigma_{n-1}(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\tau_{n+1}(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) \tag{86}$$

$$\rho_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k)\tau_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k) - \rho_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\tau_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) = \tilde{\delta}\tau_{n-1}(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\rho_{n+1}(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) \tag{87}$$

$$\tau_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\tau_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) - \tau_{n-1}(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\tau_{n+1}(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) = \sigma_n(\bar{z}_k + \mathrm{i}\tilde{\delta}^k/2k)\rho_n(\bar{z}_k - \mathrm{i}\tilde{\delta}^k/2k) \tag{88}$$

where $\tilde{\delta}$ is used instead of λ^{-2} and dependence on z_k is omitted.

Equations (81)–(83) and (86)–(88) are the main result of this paper. They present an infinite number of the DDEs of the ALH under the zero boundary conditions (46) in the form of six difference-functional equations. Thus we have derived the 'functional' representation of the ALH. Analogous results can be obtained for some other classes of boundary conditions, say, for the so-called finite-density ($q_n \to$ constant as $n \to \pm\infty$) or quasiperiodical ones. Before proceeding further I would like to mention the following problem. We have split the ALH into two subhierarchies (the 'positive' and 'negative' ones) which seems to be rather natural: one of the subhierarchies can be obtained from the other using the complex conjugation. Nevertheless, it would be interesting to obtain, instead of two sets of functional equations ((81)–(83) for the 'positive' hierarchy and (86)–(88) for the 'negative' one) one set of equations which takes into account both 'positive' and 'negative' flows. I cannot do this at present, and it will be a subject of following studies.

5. Hirota's representation of the ALH

It is already known that the D-operators calculus invented by Hirota is not only an ingenious tool for deriving some families of solutions for integrable equations. It is a convenient way of operating with integrable hierarchies, which enables us to reveal some regularities in their structure. Below are the main results in Hirota's formalism, which will demonstrate some interesting features of the ALH. In what follows I will deal only with 'positive' subhierarchy, because for the 'negative' one all results can be obtained using the complex conjugation ($\sigma_n = -\kappa\bar{\rho}_n$, etc). Using the following property of the Hirota's operators

$$\exp\{aD_z\} f(z) \cdot g(z) = f(z+a)g(z-a) \tag{89}$$

and introducing

$$D(\delta) = \sum_{k=1}^{\infty} \frac{\delta^k}{k} D_k \tag{90}$$

one can rewrite (81)–(83) as

$$\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right](\sigma_n \cdot \tau_n - \tau_n \cdot \sigma_n - \delta\sigma_{n+1} \cdot \tau_{n-1}) = 0 \tag{91}$$

$$\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right](\rho_n \cdot \tau_n - \tau_n \cdot \rho_n + \delta\tau_{n+1} \cdot \rho_{n-1}) = 0 \tag{92}$$

$$\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right](\tau_{n+1} \cdot \tau_{n-1} - \tau_n \cdot \tau_n + \sigma_n \cdot \rho_n) = 0. \tag{93}$$

One of the advantages of this viewpoint is that one can obtain an explicit form of the jth equation of the ALH, which is difficult to do in the framework of the standard IST technique discussed in the section 2. This can be done in terms of the Schur's polynomials

$$\exp\left\{\sum_{m=1}^{\infty} x^m f_m\right\} = \sum_{m=0}^{\infty} x^m \chi_m(f_1, f_2, \ldots). \tag{94}$$

(I will use below the designation $\chi_m(f_k) \equiv \chi_m(f_1, f_2, \ldots)$.) By simple calculations equations (91) and (92), which can be rewritten as,

$$2\mathrm{i}\sin\left[\frac{1}{2}D(\delta)\right]\sigma_n \cdot \tau_n = \delta\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right]\sigma_{n+1}\cdot\tau_{n-1} \tag{95}$$

$$2\mathrm{i}\sin\left[\frac{1}{2}D(\delta)\right]\rho_n \cdot \tau_n = -\delta\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right]\tau_{n+1}\cdot\rho_{n-1} \tag{96}$$

and equation (93) can be presented as

$$\left\{\chi_j\left(\frac{\mathrm{i}D_k}{2k}\right) - \chi_j\left(-\frac{\mathrm{i}D_k}{2k}\right)\right\}\sigma_n\cdot\tau_n = \chi_{j-1}\left(\frac{\mathrm{i}D_k}{2k}\right)\sigma_{n+1}\cdot\tau_{n-1} \tag{97}$$

$$\left\{\chi_j\left(\frac{\mathrm{i}D_k}{2k}\right) - \chi_j\left(-\frac{\mathrm{i}D_k}{2k}\right)\right\}\rho_n\cdot\tau_n = -\chi_{j-1}\left(\frac{\mathrm{i}D_k}{2k}\right)\tau_{n+1}\cdot\rho_{n-1} \tag{98}$$

$$\chi_j\left(\frac{\mathrm{i}D_k}{2k}\right)(\tau_{n+1}\cdot\tau_{n-1} - \tau_n\cdot\tau_n + \sigma_n\cdot\rho_n) = 0 \tag{99}$$

for $j = 1, 2, \ldots$.

It was noted in section 2 that 'positive' subhierarchy of the ALH (as well as the 'negative' one) can be presented as a hierarchy of PDEs, which can easily be derived from (81)–(83). Using the identities

$$D_1\sigma_n(z_k + \mathrm{i}\delta^k/2k)\cdot\tau_n(z_k - \mathrm{i}\delta^k/2k) = -\mathrm{i}\sigma_{n+1}(z_k + \mathrm{i}\delta^k/2k)\tau_{n-1}(z_k - \mathrm{i}\delta^k/2k) \tag{100}$$

$$D_1\tau_n(z_k + \mathrm{i}\delta^k/2k)\cdot\rho_n(z_k - \mathrm{i}\delta^k/2k) = -\mathrm{i}\tau_{n+1}(z_k + \mathrm{i}\delta^k/2k)\rho_{n-1}(z_k - \mathrm{i}\delta^k/2k) \tag{101}$$

$$D_1\tau_n(z_k + \mathrm{i}\delta^k/2k)\cdot\tau_n(z_k - \mathrm{i}\delta^k/2k) = \mathrm{i}\delta\sigma_{n+1}(z_k + \mathrm{i}\delta^k/2k)\rho_{n-1}(z_k - \mathrm{i}\delta^k/2k) \tag{102}$$

which follow from (71) and (80) for $j = 1$, equations (81)–(83) can be rewritten as

$$\hat{G}(\delta)\begin{pmatrix}\sigma\cdot\tau\\ \tau\cdot\rho\\ \sigma\cdot\rho+\tau\cdot\tau\end{pmatrix} = 0 \tag{103}$$

where σ, ρ and τ stand for σ_n, ρ_n and τ_n with n being fixed and the operator $\widehat{G}(\delta)$ is defined by

$$\hat{G}(\delta) = 2\mathrm{i}\sin\left[\frac{1}{2}D(\delta)\right] - \mathrm{i}\delta D_1\exp\left[\frac{\mathrm{i}}{2}D(\delta)\right]. \tag{104}$$

Expanding (103) in power series in δ one can obtain a hierarchy of partial differential equations

$$\hat{G}_j\begin{pmatrix}\sigma\cdot\tau\\ \tau\cdot\rho\\ \sigma\cdot\rho+\tau\cdot\tau\end{pmatrix} = 0 \qquad j = 2, 3, \ldots \tag{105}$$

where operators $\hat{G}_j$ are defined by

$$\hat{G}(\delta) = \sum_{j=2}^{\infty}\frac{\delta^j}{j}\hat{G}_j. \tag{106}$$

Some first equations of this hierarchy are ones given by (105) with

$$\hat{G}_2 = \mathrm{i}D_2 + D_{11} \tag{107}$$

$$\hat{G}_3 = \mathrm{i}D_3 + \frac{3}{4}D_{21} + \frac{\mathrm{i}}{4}D_{111} \tag{108}$$

$$\hat{G}_4 = \mathrm{i}D_4 + \frac{2}{3}D_{31} + \frac{\mathrm{i}}{4}D_{211} - \frac{1}{12}D_{1111}. \tag{109}$$

A rather interesting consequence of (81)–(83) can be obtained by excluding σ_n and ρ_n. It is straightforward to show, using (71) and (80) for $j = 1, 2$, that

$$[2D_1 - \delta(D_2 + \mathrm{i}D_{11})]\tau_n(z_k + \mathrm{i}\delta^k/2k) \cdot \tau_n(z_k - \mathrm{i}\delta^k/2k) = 0 \tag{110}$$

or, again using the $\exp[\mathrm{i}D(\delta)/2]$ operator,

$$[2D_1 - \delta(D_2 + \mathrm{i}D_{11})] \exp\left[\frac{\mathrm{i}}{2}D(\delta)\right]\tau \cdot \tau = 0 \tag{111}$$

where $\tau \equiv \tau_n$ (for any n). Expanding this equation in powers of δ one can again obtain an infinite number of equations, this time for one function, τ. The first few of them are

$$(4D_{31} - 3D_{22} + D_{1111})\tau \cdot \tau = 0 \tag{112}$$

$$(3D_{41} - 2D_{32} + D_{2111})\tau \cdot \tau = 0 \tag{113}$$

$$(96D_{51} - 60D_{42} + 20D_{3111} + 15D_{2211} - D_{111111})\tau \cdot \tau = 0. \tag{114}$$

It is interesting that equation (112) is nothing other than the KP equation. Indeed, it can be verified by straightforward (though rather cumbersome) calculations that the quantity $u = r_{n-1}p_nq_{n+1}$ for any n solves the equation

$$\partial_1(4\partial_3 u + \partial_{111}u + 12u\partial_1 u) = 3\partial_{22}u. \tag{115}$$

So, we have obtained a remarkable result: the KP equation turns out to be 'embedded' in the ALH!

6. Conclusion.

In this work a representation of the ALH in the form of difference-functional equations has been obtained. This result is interesting from several viewpoints. First, it clearly demonstrates a common origin of all equations of the hierarchy. Second, such an approach can be useful as an easy tool for generating a large number of solutions for the ALH, such as multisoliton, 'Wronskian' and some others. An interesting transformation of the results obtained arises when one considers the problem of quasiperiodic solutions. The functional relations (81)–(83) and (86)–(88) become the Fay's trisecant formulae for the θ-functions. This means that the ALH can be used to describe flows over the finite-genus Riemann surfaces and to characterize such surfaces. To my knowledge, such algebro-geometrical aspects of the ALH have not been discussed in literature.

Of more interest is the question of the 'universality' of the ALH. It is known that some equations can be 'embedded' into the ALH. It has been shown that the ALH 'contains' the 2D Toda lattice [9] (see also [13]), $\mathrm{O}(3,1)$ σ-model [14], the Davey–Stewartson (DS) equation and the Ishimori model [15]. In the last paper it was shown that the derivative nonlinear Schrödinger equation can be 'embedded' into the ALH, which implies that the same can also be done for the AKNS. In this paper I have shown that the KP equation (hierarchy) can also be composed of the ALH flows. An impression arises that the ALH possesses some kind of 'universality': almost all 'classical' hierarchies, such as the AKNS, DS, KP hierarchies, may be 'constructed' from the ALH. The results of the works [9, 13–15] (see also [16, 17]) are in some sense 'empirical' facts: they can be easily verified by simple calculations, but this does not answer the question of why such apparently different models turn out to be interrelated. The approach described above, especially the results presented in section 5, provides some insight into this problem.

It is known that an integrable system is much more than a set of solvable equations. One of the best illustrations of this thesis is the KP equation. It was derived to describe nonlinear waves in weakly dispersive media, but in the last two decades there has been an immense number of works devoted to this equation, which are far from the hydrodynamics or theory of the acoustic waves, where it originally appeared. The KP equation and its hierarchy has been considered from different viewpoints. It has been studied in the framework of the theory of Grassmannians, representation theory of the Kac–Moody algebras and it has been applied to the problem of characterization of the Jacobi varieties, etc. Returning to the ALH one has to admit that studies of this hierarchy were mostly restricted to 'practical' problems, i.e. to questions related to solving its equations. At present we have a wide range of solutions but know little about the theoretical aspects of the ALH. Having presented the ALH in the bilinear form, (81)–(83) and (86)–(88), we came close to the range of problems which arise naturally when an integrable system is considered not only as a set of equations that should be solved, but from a more general viewpoint. Comparing the functional equations obtained above with, e.g. the analogous equations for the τ-functions of the KP hierarchy one can see that the ALH is in some sense richer: instead of one τ-function we have an infinite number of triplets $\{\tau_n, \sigma_n, \rho_n\}$ (note that the last two, σ_n and ρ_n, in contrast to τ_n, are complex). Hence, one can expect that all group-theoretical and algebro-geometrical constructions revealed behind the KP hierarchy can be found, even in some extended form, in the case of the ALH. Thus, this work can be viewed as a starting point to a series of studies that will link the ALH with the theory of Grassmannian manifolds, representation theory of infinite-dimensional algebras. Another interesting problem is the operator (bosonic/fermionic) content of the ALH. Note that the question of Hamiltonian structures that can be associated with the hierarchy considered has not been touched in this paper. These problems cannot be analysed in the framework of one paper and these issues will be addressed in following investigations.

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